RIKEN–IPMU mini-workshop on QCA and K-theory: **Quantum Cellular Automata and Algebraic K-theory**

Lectures by Bowen Yang
Notes Written by Mattie Ji
Edited by Mattie Ji and Bowen Yang

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0 Introduction

The **RIKEN-IPMU mini-workshop on QCA and K-theory** was a joint workshop held at the Kavli Institute for the Physics and Mathematics of the Universe (IPMU) in the University of Tokyo from August 24th to August 26th, 2026. The workshop was organized by Kantaro Ohmori, Shuhei Ohyama, Mayuko Yamashita, and Yuji Tachikawa, and featured a three-day lecture series by Bowen Yang. For more information, please visit the conference website here.

The goal of the workshop is to introduce quantum cellular automata (QCA) and algebraic K-theory, and to explain recent developments connecting the two subjects. The workshop focuses on the following papers:

1. *Categorifying Clifford QCA* by Bowen Yang [Yan26a]
2. *Quantum Cellular Automata: The Group, the Space, and the Spectrum* by Mattie Ji and Bowen Yang [JY26b]
3. *K-Theoretic Obstructions to Linearizing QCA Representations* by Mattie Ji and Bowen Yang [JY26a]

These notes were originally live-texed by Mattie Ji, with additional editing works done by Mattie Ji and Bowen Yang after the conference concluded. The recordings for all 3 lectures are also available on YouTube at the following link.
We thank the organizers and the scientific committee for organizing the workshop, and gratefully acknowledge the financial support provided for our travel and participation at this workshop.

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1 Lecture 1 (August 24th, 2026)

The first half of the first lecture will give a personal take for why (generalized) cohomology theories from homotopy theory inevitably shows up in the setting of quantum cellular automata (QCA). This framework should work for a range of classification problems in topological phases.

1.1 Features of Topological Phases (from the Lattice Perspective)

There are many parallel themes between physics and mathematical concepts in the study of topological phases. In this section, we will use blue for physics and red for mathematics.

1. To study any sort of physical systems, we need the notion of locality (e.g., local Hamiltonians). To make sense of locality, we need some mathematical notion of distance, i.e., metric spaces.
2. We also want to classify them into phases to gain systematic understandings. For two systems in the same phase, they may be related by a path of systems of the same type between them. This is a concept called homotopy.
3. These phases are often monoidal, and the classification has the ability to do what is called stacking of phases. This should correspond to the notion of a monoid or abelian group.
4. (A slightly contentious idea): To study phases or models, we need to get to a thermodynamic limit, which corresponds to the idea of an unbounded metric space by taking things to infinity.
5. (Will Not Be Featured to Prominently) A topological phase should somehow be "robust against dirt" (e.g., your lattice may not be exactly $\mathbb{Z}^n$ but perturbed a bit). In math, there is a theory called coarse geometry that study these kind of phenonmenon. For us, this just means we view an n-dimensional system as $\mathbb{Z}^n$, or some discrete "well-spreadout" metric subspace of $\mathbb{R}^n$, and that $\mathbb{R}^n$ is so-called coarsely equivalent to $\mathbb{Z}^n$.

To summarize the themes, we have that

Physics	**Mathematics**
Locality (e.g., local Hamiltonian)	Metric Space
Classification to Phases	Homotopy
Stacking	Monoid or Abelian Group
Thermodynamic Limit	Unbounded Metric Space
Robust Against Dirt	Coarse Geometry

1.2 Open Cone Construction and Proto-Theorems for Topological Phases of Matter

The topic of this minicourse is on **quantum cellular automata**, which is a topic in quantum physics, and **algebraic K-theory**, which is a topic in **stable homotopy theory** and is a way to produce **generalized (co)homology theories**. Before we dive into the two specific topics, there is a fundamental question we should justify first.

Question 1.1. Why does generalized (co)homology theories show up in quantum physics?

Here we explain one motivation for this in the subject of **topological phases of matter**. From the Table above, we know we ideally would like to work with unbounded metric spaces, whereas generalized (co)homology theories tend to be determined on **finite CW complexes**, which are inherently not unbounded.

The open cone construction is a way to connect

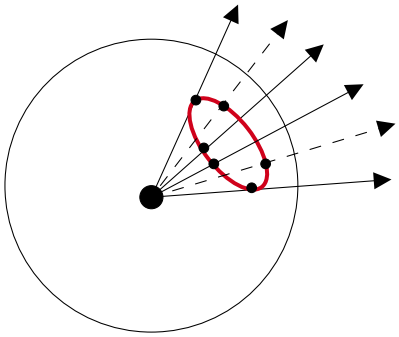
$$\text{finite CW complexes} \mapsto \text{unbounded metric spaces},$$

thereby giving a bridge between the two subjects (which we will elaborate).

Definition 1.2. Let $X \subseteq S^n$ be a finite CW complex in S^n, and S^n is the unit sphere in $\mathbb{R}^{n+1}$, the Open Cone Functor assigns

$$O : X \subseteq S^n \mapsto O(X) \coloneqq \{t \cdot x \in \mathbb{R}^{n+1} \mid t \in [0, \infty), x \in X\}$$

The following is a picture of the open cone construction on a subcomplex of S^2:



Example 1.3. Take $X = S^n$, then $O(S^n) = \mathbb{R}^{n+1}$.

We make the following observation, which is also a fun exercise.

Lemma 1.4. Let $X * Y$ denote the join of X and Y, then

$$O(X * Y) = O(X) \times O(Y).$$

In particular, the suspension of X can be obtained as the join of X and S^0, and hence we have:

Corollary 1.5. We have the equality

$$O(\Sigma X) = O(X) \times O(S^0) = O(X) \times \mathbb{R}^1.$$

With all of these setups, there is a **proto-theorem** of what we would expect to show up for any kind of topological phases problems:

1. Let X be a finite CW complex.
2. Consider the open cone $O(X)$, which is itself an **unbounded metric space**.
3. We seek a space $K(O(X))$, which should be thought of as a moduli space of interests, which is where "the physics is happening"
4. From the space $K(O(X))$, we can produce its homotopy groups $\{\pi_i K(O(X))\}_{i \geq 0}$.
5. The hope is that this is a generalized homology theory for finite CW complexes.

$$X \xrightarrow{\text{Generalized Homology Theory}} O(X) \longrightarrow K(O(X)) \longrightarrow \{\pi_i K(O(X))\}_{i \geq 0} \ .$$

We are not saying that every question in topological phases would fit into a question like this, but this is a reasonable / intuitive enough setting where many questions would arise like this. This is a prototype for a theorem of how to set-up a problem for establishing this is a generalized homology theory (this is like a question you look for).

Remark 1.6. The physical meaning of X would be the boundary at ∞. For many applications, this X would be just S^n. Physically, an n-dimensional Euclidean space is the space (not the time), like a piece of the material.

Temporary Notation: Let us call $h_i(X) \coloneqq \pi_i K(O(X))$.

Having to come up with a space for every CW complex is quite difficult. Let us restrict to perhaps the case of spheres.

A very important property of generalized homology theory is that there is a **canonical isomorphism** of the form:

$$h_{i+1}(\Sigma X) \cong h_i(X).$$

Plugging in $X = S^{n-1}$, we will have that

$$\pi_{i+1}(K(\mathbb{Z}^{n+1})) \cong h_{i+1}(S^n) \cong h_i(S^{n-1}) \cong \pi_i K(\mathbb{Z}^n).$$

Remark 1.7. Here we wrote $K(\mathbb{Z}^{n+1})$ instead of $K(\mathbb{R}^{n+1})$ due to our early remark on coarse equivalence.

In other words, we should have the equivalence between

$$\Omega K(\mathbb{Z}^{n+1}) \simeq K(\mathbb{Z}^n),$$

where $\Omega(-)$ denotes the (based) loop space.

Therefore, from this **proto-theorem**, we get a list of spaces

$$\{K(\mathbb{Z}^n)\}_{n\geq 0}, \text{ with the data of deloopings } K(\mathbb{Z}^n) \simeq \Omega K(\mathbb{Z}^{n+1}).$$

This data is an example of what is called an Ω-spectrum.

These two proto-theorems have been realized in practice, and we will explain them in subsequent lectures.

Theorem 1.8 (Y. [Yan26a])**.** In the paper *Categorifying Clifford QCA* (Communications in Mathematical Physics, CMP 2026), this proto-theorem holds for Clifford QCAs.

Theorem 1.9 (Ji-Y. [JY26b])**.** In arXiv `2602.16572`, this proto-theorem holds for all general QCAs (and developed in an arbitrary algebraic setting over a commutative ring R with unity).

Furthermore, in a follow-up work [JY26a] to [JY26b], we used the space/spectrum-level constructions for general QCAs to determine anomalies of (0-form) symmetries on quantum systems.

1.3 Introduction to Algebraic K-Theory

Let us switch gears to talk about what K-theory is.

Question 1.10. What is the enterprise of algebraic K-theory?

Idea: K-theory is a general methodology for linearizing classification questions.

There are various kinds of classification questions that appear in life.

1.3.1 The K_0-Group

1. Topology: In topology, we can classify real vector bundles $\mathrm{Vect}_{\mathbb{R}}(X)$ and complex vector bundle $\mathrm{Vect}_{\mathbb{C}}(X)$. This is quite hard to compute.

For two vector bundles $E \to X, F \to X$, it makes sense to add them as Whitney sums, but it makes a lot less sense to subtract them. What we can do is to **formally impose minus signs** to get what is called their **group completions**: $\mathrm{Vect}_{\mathbb{R}}(X)^{gp}, \mathrm{Vect}_{\mathbb{C}}(X)^{gp}$. This is called (zeroth) **topological K-theory**, which is easier to compute.

2. Algebra: Take R to be your favorite ring, we can try to classify finitely generated projective R-modules, and again it makes sense to add modules but less to subtract them, so we formally add inverses to give

$$K_0(R) = \mathrm{Proj}_R^{gp}.$$

3. Symmetric Monoidal Category: Let $(\mathcal{C}, \otimes)$ be a small monoidal category, and consider $M_{\mathcal{C}}$ to be the isomorphism classes of objects. Then we define

$$K_0(\mathcal{C}) = M_{\mathcal{C}}^{gp}.$$

To be more formal, the K_0 is defined as follows.

Definition 1.11. For $(M, +)$ a commutative monoid, its **group completion** is the free abelian group on elements in M with certain relations as

$$M^{gp} = \mathbb{Z}[M]/\langle [m] + [n] = [m+n], \forall m, n \in \mathbb{Z} \rangle.$$

This has the **universal property** that for any commutative monoid homomorphism $\phi : M \to A$ into an abelian group A, there is a unique factorization

$$\begin{array}{ccc} M & \xrightarrow{m \mapsto [m]} & M^{gp} \\ {\scriptstyle\phi}\downarrow & \swarrow_{\exists!} & \\ A & & \end{array}.$$

Example 1.12. We have the following two examples:

1. If M is $(\mathbb{Z}_{\geq 0}, +)$ with monoidal unit 0, then $M^{gp} = (\mathbb{Z}, +)$.

   When we take R to be a field $\mathbb{F}$, every finitely generated projective $\mathbb{F}$-module is free. Consequently, this shows us that
   $$K_0(\mathbb{F}) = (\mathbb{Z}_{\geq 0}, +)^{gp} = \mathbb{Z}.$$

2. If $M = (\mathbb{Z}_{>0}, \times)$ with monoidal unit 1, then $M^{gp} = (\mathbb{Q}_{>0}, \times)$.

   Let $\mathrm{Mat}(\mathbb{C})$ be the symmetric monoidal category of matrix rings over $\mathbb{C}$ with the operation of tensor products. Observe that $M_{\mathcal{C}} \cong (\mathbb{Z}_{>0}, \times)$, and hence
   $$K_0(\mathrm{Mat}(\mathbb{C})) = \mathbb{Q}_{>0}.$$

1.3.2 The K_1-Group

The definition of K_0 is not the full story, there are higher K_i's that can be defined. Let us explain the definition of $K_1(R)$, which will be relevant for QCAs later.

Definition 1.13. Let $i_n : \mathrm{GL}_n(R) \to \mathrm{GL}_{n+1}(R)$ denote the inclusion map

$$A \in \mathrm{GL}_n(R) \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} \in \mathrm{GL}_{n+1}(R).$$

We take $\mathrm{GL}(R)$ as the colimit of $\mathrm{GL}_n(R)$ with respect to i_n. From here, we have:

$$K_1(R) \coloneqq \mathrm{GL}(R)^{ab}.$$

Example 1.14. When $R = \mathbb{F}$ is a field, we have $K_1(\mathbb{F}) = \mathbb{F}^\times$ realized by the determinant map.

In quantum physics, however, we might be motivated to take tensor products instead.

Question 1.15. Consider the map $\phi_{n,rn} : \mathrm{GL}_n(R) \to \mathrm{GL}_{r\cdot n}(R)$ by sending

$$A \mapsto A \otimes I_r,$$

where $A \otimes I_r$ is the block diagonal matrix with r-copies of A.

We can consider

$$\mathrm{GL}_\otimes(R) \coloneqq \mathrm{colim}_{n\in\mathbb{Z}_{>0}} \mathrm{GL}_n(R),$$

where the colimit is taken to be the divisibility poset with transition map being ϕ's described above.

Theorem 1.16 (Bass [Bas68])**.** We have that

$$\mathrm{GL}_\otimes(R)^{ab} \simeq K_1(R) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

1.3.3 Hints towards the Higher Algebraic K-Theory Groups

We shouldn't be satisfied with just getting these groups. We would like an actual space. In algebraic K-theory, the construction is done as follows.

In general, it is possible to define the groups K_i for higher i. Roughly, $K_i(\mathcal{C})$ are the homotopy groups π_i of a space $K(\mathcal{C})$ (called the K-theory space of $\mathcal{C}$). The space $K(\mathcal{C})$ is formed from a topological group-completion that generalize K_0. More details will be discussed in later lectures (Section 2.1). We also recommend the textbook [Wei13] for more details on algebraic K-theory.

Let us specialize to **two examples of interest**.

Definition 1.17. Let R be a ring and Proj_R be the category of (finitely generated) projective R-modules, and $\mathrm{Proj}\, R_{\neq 0}$ be the category of (finitely-generated) non-zero projective R-modules. We can consider two symmetric monoidal structures $(\mathrm{Proj}\, R, \oplus)$ and $(\mathrm{Proj}\, R_{\neq 0}, \otimes)$ by direct sum and tensor product.

There is a process called **group completion** that produces infinite loop spaces. From here, we define:

$$K(R) \coloneqq (\mathrm{Proj}\, R, \oplus)^{gp} \text{ and } K_\otimes(R) \coloneqq (\mathrm{Proj}\, R, \otimes)^{gp}.$$

Furthermore, $K_i(R) \coloneqq \pi_i K(R)$ matches the previous definition for $i = 0, 1$.

We note Theorem 1.16 can be equivalently restated as that $\pi_1(K_\otimes(R)) \cong K_1(R) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Theorem 1.18 (Bass-May-Weibel)**.** It is in fact true that, for $i \geq 1$,

$$\pi_i K_\otimes(R) \simeq \pi_i K(R) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

The case for $i = 1$ is due to Bass in [Bas68]. The more general case was implicit in May's textbook [May77] and was pointed out by Weibel in [Wei81a, Wei81b].

Our discussion doesn't have much physics yet, apart from talking about tensor products, but the next section we introduce would be more illuminating on the collections.

1.4 Bounded K-Theory

There is a notion called **bounded K-theory**, which was invented by Pedersen and Weibel [PW85] to find a model for negative K-theory.

We will not talk about this original motivation, however, but will explain this from a lattice model perspective.

Let (Λ, d) be a lattice with a metric (typically take $\Lambda = \mathbb{Z}^n$), and R be a ring.

Definition 1.19. The symmetric monoidal category $\mathcal{C}_\Lambda(R)$ has:

1. Objects being $\bigoplus_{x\in\Lambda} R^{m_x}$ for some function $m : \Lambda \to \mathbb{N} = \{0, 1, 2, ...\}$. We can think of this as placing a finitely generated free R-module on each lattice point $x \in \Lambda$.
2. Morphisms are R-linear maps
$$\varphi : \bigoplus_{x\in\Lambda} R^{m_x} \to \bigoplus_{y\in\Lambda} R^{\ell_y}$$
Write $\varphi_{yx} : R^{m_x} \to R^{\ell_y}$ as the specific part happening between x and y. We require that there exist some $r > 0$ such that
$$\varphi_{yx} = 0 \text{ whenever } d(x, y) > r.$$
These are morphisms that are **uniformly bounded in propagation**.
3. With the monoidal structure of pointwise direct sums.

This is a symmetric monoidal category, and we can talk about its group-completion. We will only talk about its homotopy groups, but it turns out that

$$K_i(\mathcal{C}_{\mathbb{Z}^n}(R)) \simeq K_{i+1}(\mathcal{C}_{\mathbb{Z}^{n+1}}(R)), i \geq 1.$$

Remark 1.20. For $n = 0$, we have the usual K-theory of R, but we now have these other groups coming out, and we get what is called the **non-connective algebraic K-theory of R.**

Pedersen and Weibel probably were not thinking about condensed matter physics when writing this paper, but this setting looks very familiar! The only difference is that we should replace direct sums to tensor products, R^n to $M_n(R)$, and morphisms to locality preserving isomorphisms (which include **quantum cellular automata**).

Remark 1.21. Although Pedersen and Weibel did not study topological phases of matter, they later used open cone constructions to make generalized (co)homology theories for bounded K-theory in [PW89].

1.5 Algebraic L-theory

One reason the speaker likes about K-theory is due to its close relation with linear algebra. Quantum physics is also close to linear algebra. One thing we do in linear algebra is to consider forms

$$\Omega : R^n \times R^n \to R.$$

One example could be when n is even, and we consider symplectic forms, for example,

$$J : R^{2n} \times R^{2n} \to R, J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}.$$

The symplectic group $\mathrm{Sp}_{2n}(R)$ are those preserving the symplectic forms. We can consider the group

$$\mathrm{Sp}(R) \coloneqq \mathrm{colim}_n \mathrm{Sp}_{2n}(R).$$

It turns out this is clsoely related to the story of algebraic L-theory.

Definition 1.22. The 3rd algebraic L-theory is defined as $L_3(R) \coloneqq \mathrm{Sp}(R)^{ab}/\langle J \rangle$.

More generally, one is able to definee a space $L(R)$ such that

$$\pi_3 L(R) = L_3(R),$$

and define other groups. This is called **algebraic L-theory**. This is a lot easier to compute than algebraic K-theory, as it happens to have 4-periodicity.

It turns out we can massage Clifford QCA's (which will be defined later in Section 2.5) into the exact bounded K-theory framework, which will be explained tomorrow.

2 Lecture 2 (August 25th, 2026)

2.1 Q&A From Last Time: More Formal Definition of Algebraic K-Theory

Las time, we did not clarify what group-completion really meant. Here we first explain what they are.

To generalize the definition of K-theory to higher degree, one better define an actual space. There are many different ways to do this by Quillen, Waldhausen, Segal, etc. The version we will focus on is **Segal's K-theory**, which is a **topological version of the algebraic group completion procedure** mentioned.

Here we give a somewhat informal definition.

Definition 2.1. Let $\mathcal{C}$ be a (small) symmetric monoidal category, the classifying space $B\mathcal{C}$ inherits the tensor product operation and is a homotopical version of a topological monoid (with some coherence data).

The K-theory space $K(\mathcal{C})$ is a homotopical version of a topological group (with some coherence data) and a map (called **group completion**) $\iota : B\mathcal{C} \to K(\mathcal{C})$ such that every coherent multiplicative map $\phi : B\mathcal{C} \to Y$ factors through as

$$\begin{array}{ccc} B\mathcal{C} & \xrightarrow{\iota} & K(\mathcal{C}) \\ {\scriptstyle\phi}\downarrow & \swarrow & \\ Y & & \end{array}$$

Here we no longer require the lift to be unique, but there is a **contractible space** of choices of such lifts. Furthermore, taking π_0 recovers the usual algebraic group completion, and the construction is functorial to lax symmetric monoidal functors.

We define the higher K-theories of $\mathcal{C}$ as the homotopy groups of $K(\mathcal{C})$,

$$K_i(\mathcal{C}) \coloneqq \pi_i(K(\mathcal{C})).$$

When $\mathcal{C} = \mathrm{Proj}(R)$, we write the higher algebraic K-theories of R as

$$K_i(R) \coloneqq \pi_i(K(\mathrm{Proj}(R)).$$

Up-Shot: K-theory is a way to homotopically group-complete.

Remark 2.2. More formally, $B\mathcal{C}$ is an example of what is called an $\mathbb{E}_\infty$-space (also called an $\mathbb{E}_\infty$-monoid), and $K(\mathcal{C})$ is an example of what is called a group-like $\mathbb{E}_\infty$-space (also called an $\mathbb{E}_\infty$-group). We use $\mathrm{CMon}(\mathcal{S})$ and $\mathrm{CGrp}(\mathcal{S})$ to denote the $\mathbb{E}_\infty$-monoids and $\mathbb{E}_\infty$-groups.

The group-completion described above is left-adjoint to the forgetful functor $\mathrm{CGrp}(\mathcal{S}) \to \mathrm{CMon}(\mathcal{S})$ (see Corollary 4.4 and Remark 4.5 of [GGN15] for example) and there is thus an equivalence

$$\mathrm{Map}_{\mathrm{CGrp}(\mathcal{S})}(K(\mathcal{C}), G) \simeq \mathrm{Map}_{\mathrm{CMon}(\mathcal{S})}(B\mathcal{C}, G),$$

which describes the universal property given in Definition 2.1

The following is a far more concrete definition of group completion and $K(\mathcal{C})$.

Definition 2.3. Let $\mathcal{C}$ be a (small) symmetric monoidal category, its classifying space $B\mathcal{C}$ has the natural structure of a homotopy commutative, homotopy associative H-space. $B\mathcal{C}$ is essentially a monoid up to homotopy,

and this makes the following two structures well-defined:

1. $\pi_0(B\mathcal{C})$ is a commutative monoid.
2. For any commutative ring R, the homology $H_*(B\mathcal{C}; R)$ is a ring[a].

The **K-theory space** $K(\mathcal{C})$ is defined as the **group completion** $\iota : B\mathcal{C} \to K(\mathcal{C})$ of $B\mathcal{C}$, which is characterized by the following property

1. $K(\mathcal{C})$ is am H-space and $\pi_0(K(\mathcal{C}))$ is a group (This is sometimes called a **group-like** H**-space**.
2. The induced map $\iota_* : \pi_0(B\mathcal{C}) \to \pi_0(K(\mathcal{C}))$ is **the algebraic group completion** of the commutative monoid $\pi_0(B\mathcal{C})$.
3. The homology of $H_*(K(\mathcal{C}); R)$ is **the localization** of $H_*(B\mathcal{C}; R)$ at $\pi_0(B\mathcal{C})$ for any commutative ring R. Here, the homologies are rings because the relevant spaces are H-spaces.

The i-th K-theory of $\mathcal{C}$ is denoted $K_i(\mathcal{C}) \coloneqq \pi_i(K(\mathcal{C}))$.

[a] This is sometimes called the **Pontryagin ring**.

Remark 2.4. If $B\mathcal{C}$ is a topological monoid (e.g., when $\mathcal{C}$ is strictly unital, strictly associative, strictly commutative, etc.), then the McDuff-Segal group completion theorem [MS76] shows that $K(\mathcal{C}) \cong \Omega B(B\mathcal{C})$.

2.2 Quantum Cellular Automata (QCA)

We hinted at the definition of QCAs at the end of last lecture, and here we will formally define what they are.

Notation: We use the following notation:

1. R is any commutative ring with identity.
2. $M_n(R)$ is the ring of $n \times n$ matrices over R.

Definition 2.5. Let X be a metric space (e.g., $X = \mathbb{Z}^n, n \geq 0$) and $q : X \to \mathbb{Z}_{>0}$ be a set function.

1. We say q is a **quantum spin system** if $q(x) = 1$ outside of a locally finite set on X (for example if $X = \mathbb{Z}^n$, a quantum spin system would be any set function).
2. Let S be a finite subset of X, we define $\mathcal{A}(S, q) \coloneqq \bigotimes_{x \in S} M_{q(x)}(R)$. We now define the **algebra of observables** over (X, q) as
$$\mathcal{A}(X, q) \coloneqq \operatorname{colim}_{S \subset_{\text{finite}} X} \mathcal{A}(S, q),$$
where for $S \subseteq T$ finite subsets, the map $\mathcal{A}(S, q) \to \mathcal{A}(T, q)$ is given by sending
$$A \mapsto A \otimes \mathrm{id}_{T \setminus S}\,.$$

An element in $\mathcal{A}(X, q)$ should be thought of as a finite sum of finite tensors of matrices.

Remark 2.6. When $R = \mathbb{C}$, $\mathcal{A}(X, q)$ has a $*$-algebra structure coming from an adjoint operation.

Definition 2.7. For any single element $A \in \mathcal{A}(X, q)$, then A itself must be contained in some finite subalgebra.

From here, we define the support of A as

$$\operatorname{supp} A = \bigcap_{S \subset_{\text{finite}} X, A \in \mathcal{A}(S,q)} S.$$

Remark 2.8. It may be more advantageous to think of $\mathcal{A}(X,q)$ not as some global object, but as some kind of co(pre)sheaf valued on subsets.

Now we may finally define what is **quantum cellular automaton** is.

Definition 2.9. A QCA α is a **locality-preserving automorphism** of the algebra $\mathcal{A}(X,q)$. To unpack this, this means:

1. α is a map $\mathcal{A}(X,q) \to \mathcal{A}(X,q)$ that is an R-algebra isomorphism.
2. There exists some $\ell \geq 0$, that only depends on α, such that for any $A \in \mathcal{A}(X,q)$,

$$\operatorname{supp} \alpha(A) \subseteq \text{the } \ell\text{-neighborhood of } \operatorname{supp} A.$$

When $R = \mathbb{C}$, α is called **unitary** if α commutes with the adjoint operation, that is

$$\alpha(A^*) = \alpha(A)^*.$$

We write $\mathcal{Q}(X,q)$ to denote the group of QCAs on $\mathcal{A}(X,q)$. Similarly, we write $\mathcal{Q}^*(X,q)$ to denote the group of all unitary QCA on $\mathcal{A}(X,q)$.

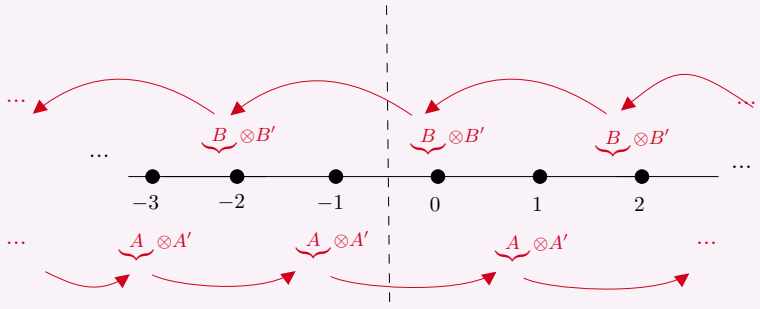
Remark 2.10. If you think of $\mathcal{A}(-,q)$ as a co(pre)sheaf, α is not exactly a morphism of co(pre)sheaves.

Here are two examples of QCA, which are not very physical but conceptually illuminating:

Example 2.11. Suppose we have a quantum spin system $q : \mathbb{Z}^1 \to \{1,2,3,...\}$ that is constant on the even numbers (with value n) and constant on the odd numbers (with value m). Fix tensor products

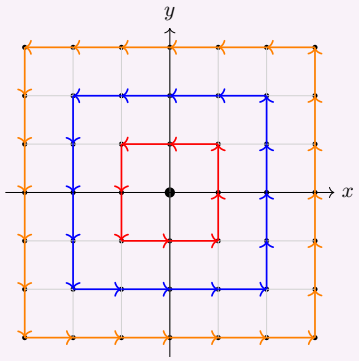
$$M_n(R) = B \otimes_R B' \text{ and } M_m(R) = A \otimes_R A'.$$

We can define a QCA by transpoting the tensor factors (that sort of looks like a Hilbert hotel) as follows:



This has spread $\ell = 2$.

Example 2.12. Suppose we have a quantum spin system $q : \mathbb{Z}^2 \to \{1, 2, 3, ...\}$ that is just a constant function. We can form a QCA via annular shifts as follows:



This has spread $\ell = 1$.

For the sake of concreteness, let us examine the definition over the case when X is a single point.

Example 2.13. When $X = *$ is a single point, we have that:

1. The collection of quantum spin system on X is the poset of positive integers, ordered by divisibility (i.e., $r \leq s$ if r divides s).
2. For a fixed quantum spin system q with value n, $\mathcal{Q}(*, q) = \mathrm{Aut}_R(M_n(R))$. If R is a field, local ring, are more generally a ring where automorphisms of matrix algebras are all inner (i.e., for fields this is called the **Skolem-Noether theorem**), then
$$\mathrm{Aut}_R(M_n(R)) \cong \mathrm{PGL}_n(R),$$
where $\mathrm{PGL}_n(R)$ stands for the $n \times n$ projective general linear matrices over R.
3. The total QCA group is
$$\mathcal{Q}(*; R) = \mathrm{colim}_{n \in \mathbb{Z}_{>0}} \mathrm{Aut}_R(M_n(R)).$$
When R is a field, local ring, or more generally a ring where automorphisms of matrix algebras are all inner, then
$$\mathcal{Q}(*; R) = \mathrm{PGL}(R) \coloneqq \mathrm{colim}_{n \in \mathbb{Z}_{>0}} \mathrm{PGL}_n(R),$$
where the colimit is indexed over the poset $(\mathbb{Z}_{>0}, |)$ under divisibility, and for d dividing n, the map $\mathrm{PGL}_d(R) \to \mathrm{PGL}_n(R)$ is given by tensoring with the identity matrix of dimension n/d.

Example 2.14. When X is a finite set with discrete metric, by cofinality of colimtis, we will have that
$$\mathcal{Q}(X; R) \cong \mathcal{Q}(*; R).$$

2.3 Quantum Circuits

Here are some more physically illuminating examples, which are known as **quantum circuits**.

Definition 2.15. α is a **single-layer quantum circuit** if there is a partition into uniformly bounded pieces:

$$X = \bigsqcup_j X_j, \quad \sup_j \operatorname{diam} X_j < \ell$$

such that

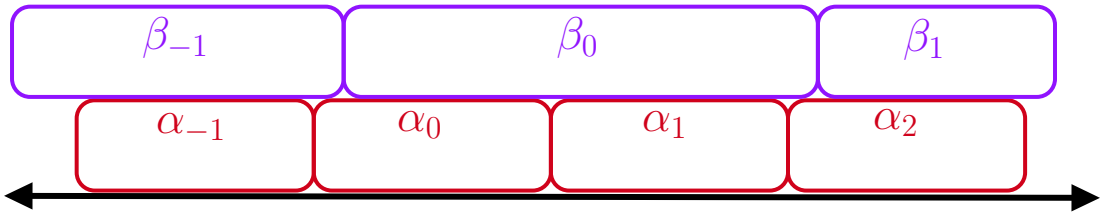
$$\alpha = \bigotimes_i \alpha_i, \text{ where each } \alpha_i \in \operatorname{Aut}_R(\mathcal{A}(X_i, q)).$$

The subgroup of $\mathcal{Q}(X, q)$ generated by single-layer quantum circuits is denoted $\mathcal{C}(X, q)$.

Here is a picture of a single-layer quantum circuit on $\mathbb{Z}^1$.



Single-layer circuits look simple, but the group they generate are quite different. For two single-layer circuits α and β, they need not share the same partition and can stack together like:



It should not come as a surprise that $\mathcal{C}(X, q)$ is often not all of $\mathcal{Q}(X, q)$. For example, the translation example would almost never be a circuit.

So far, our $\mathcal{Q}(X, q)$ and $\mathcal{C}(X, q)$ both still depend on q. We want to get rid of the q to focus on a single collection. We can do this via **stabilization**.

Definition 2.16. We define the **total QCA group over** (X, R) **as**

$$\mathcal{Q}(X; R) \coloneqq \operatorname{colim}_{\text{quantum spin systems } q} \mathcal{Q}(X, q)$$

Similarly, we can also define the **total circuit group over** (X, R) **as** $\mathcal{C}(X; R)$.

In the unitary case over $\mathbb{C}$ (when we restrict to $*$-preserving QCAs), we denote the **total unitary QCA group** and the **total unitary circuit group** as $\mathcal{Q}^*(X)$ and $\mathcal{C}^*(X)$ respectively.

To explain the colimit system, we say $q \leq r$ if $q(x)$ divides $r(x)$ for all $x \in X$, and we take the pointwise tensor product with $I_{q(x)/r(x)}$ across the arrow $q \leq r$.

Remark 2.17. In [JY26b], we also discussed other types of circuits where we take α_j to be in certain subgroups of $\operatorname{Aut}_R(M_n(R))$, as opposed to the whole group. However, over $\mathbb{C}$, the ones we looked at all the generate the same total circuit group.

Theorem 2.18 (Freedman-Hastings [FH20], Ji-Yang [JY26b])**.** The following properties hold:

1. The group $\mathcal{C}(X; R)$ is a normal subgroup of $\mathcal{Q}(X; R)$ with $\mathcal{Q}(X; R)/\mathcal{C}(X; R)$ is abelian (under nice

assumption of X).

2. If $R = k$ is algebraically closed, then

$$\mathcal{Q}(X;R)/\mathcal{C}(X;R) \text{ is the abelianization of } \mathcal{Q}(X;R).$$

3. For arbitrary R, if $X = Y \times \mathbb{Z}$ as a metric space (with the L^∞-norm), then

$$\mathcal{Q}(X;R)/\mathcal{C}(X;R) \text{ is the abelianization of } \mathcal{Q}(X;R).$$

2.4 Results in Studying QCA

The **QCA classification problem** studies the quotient $\mathcal{Q}(X;R)/\mathcal{C}(X;R)$ (i.e., classify QCAs up to circuits).

We want to pursue the computation of the group $\mathcal{Q}(X;R)/\mathcal{C}(X;R)$. Let us look at what they are over lattice systems.

1. One-Dimensional Case $X = \mathbb{Z}^1$:

Theorem 2.19 (Gross-Nesme-Vogts-Werner (GNVW) [GNVW12])**.** When $R = \mathbb{C}$ and $X = \mathbb{Z}$, then there is an isomorphism of abelian groups

$$\mathcal{Q}^*(\mathbb{Z}^1)^{ab} \cong (\mathbb{Q}_{>0}, \times).$$

This has been generalized in [JY26b] to arbitrary rings R. In these cases, they are intricately tied to what are called **Azumaya algebras**.

Definition 2.20. An R-algebra A is an **Azumaya R-algebra** if

1. The underlying R-module structure on A is finitely generated, projective, and faithful.
2. There is an R-algebra B satisfying (1) above such that

$$A \otimes_R B \cong M_n(R).$$

There is a symmetric monoidal category $\mathrm{Az}(R)$ of Azumaya R-algebras, which morphisms as R-algebra isomorphisms, and the operation is tensor product.

The generalization of the GNVW theorem by us is as follows.

Theorem 2.21 (Ji-Yang [JY26b])**.** There is an isomorphism

$$\mathcal{Q}(\mathbb{Z}^1;R)^{ab} \simeq K_0(\mathrm{Az}(R)),$$

where the right hand side is the Grothendieck group of the symmetric monoidal category of Azumaya R-algebras with tensor product.

Corollary 2.22. In particular, if $R = k$ is a field, we have that

$$\mathcal{Q}(\mathbb{Z}^1;k)^{ab} \cong \mathbb{Q}_{>0} \oplus \mathrm{Br}(k),$$

where $\mathrm{Br}(k)$ stands for the Brauer group of the field k.

We call the part of $\mathcal{Q}(\mathbb{Z}^1; R)$ coming from the Brauer group as the **Brauer index**.

2. Two-Dimensional Case $X = \mathbb{Z}^2$:
In the case of 2D, Freedman-Hastings has the following theorem. Appendix B of [Haa23] contains a reproof of the same theorem.

Theorem 2.23 (Freedman-Hastings [FH20])**.** $\mathcal{Q}^*(\mathbb{Z}^2)^{ab} = \mathcal{Q}(\mathbb{Z}^2; \mathbb{C})^{ab} = 0$.

3. Three-Dimensional Case $X = \mathbb{Z}^3$:

Conjecture 2.24. It has been conjectured that the group $\mathcal{Q}^*(\mathbb{Z}^3)^{ab}$ is infinite.

Remark 2.25. In joint work with Jeffrey Kwan and David Long, Jeongwan Haah announced in 2026 that $\mathcal{Q}^*(\mathbb{Z}^3)^{ab}$ is non-trivial by finding a non-trivial element of order 2 in it [Haa26].

Since the workshop concluded, Kwan-Long-Haah posted the paper [KLH26]. Later, Bowen Yang (the speaker) proved that $\mathcal{Q}^*(\mathbb{Z}^3)^{ab}$ is not a finitely generated abelian group in [Yan26b].

2.5 Clifford QCA

Computing these QCA classification groups seem challenging, so we can look at a simpler class of QCAs called **Clifford QCA**.

Let us now impose some simplifying assumption to define Clifford QCA.

Assumption: We assume that:

1. The quantum spin system takes the form $q(x) = d^{m(x)}$ for $m : X \to \mathbb{Z}_{\geq 0}$ and d is some fixed number.
2. This result forces $M_d(\mathbb{C})$ as only the building blocks (this is analogous to only looking at the d-primary part). By Chinese Remainder Theorem, however, it suffices to take d to be the power of a prime, but not typically as a literal prime.
3. We will also only focus on the unitary case.

With all these explicit simplifications, we now have that

$$\mathcal{A}(X, q) = \bigotimes_{x \in X} M_{d^{m(x)}}(\mathbb{C}) = \bigotimes_{x \in X} \Big(\bigotimes_{j=1}^{m(x)} M_d(\mathbb{C}) \Big).$$

We may decompose $M_d(\mathbb{C})$ even further using the following theorem.

Theorem 2.26. The algebra $M_d(\mathbb{C})$ is generated by two elements X and Z (called the clock and shift operators). Explicitly

$$X = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix} \text{ and } Z = \begin{pmatrix} 1 & & & \\ & \xi & & \\ & & \xi^2 & \\ & & & \ddots \\ & & & & \xi^{n-1} \end{pmatrix}, \quad \xi = \exp(2\pi i/d).$$

This has the property that

$$XZ = \xi^{-1} ZX \text{ and } X^d = Z^d = \text{identity}.$$

Definition 2.27. We define $\mathcal{P}(X; m)$ as the subalgebra of $\mathcal{Q}(X, d^m)$ generated by elements:

$$\mathcal{P}(X; m) \coloneqq \langle X_{x_{ij}}, Z_{x_{ij}}, \lambda : (x, j) \text{ are the labels and } \lambda \in U(1) \rangle,$$

where $x \in X$ and $1 \leq j \leq m(x)$.

Definition 2.28. We say $\alpha \in \mathcal{Q}(X, d^m)$ is **Clifford** if

$$\alpha(\mathcal{P}(X, m)) = \mathcal{P}(X, m).$$

Proposition 2.29. The restriction of α to $\mathcal{P}(X, m)$ (i.e., the data $\alpha|_{\mathcal{P}(X,m)}$ fully determines the QCA.

Proof. This is defined on the generators and α is multiplicative. ∎

We also make the following algebraic observation.

Lemma 2.30. The center of $\mathcal{P}(X, m)$ is $U(1)$, and the quotient

$$\mathrm{P}(X, m) \coloneqq \mathcal{P}(X, m) / \operatorname{Center}(\mathcal{P}(X, m)) \cong \bigoplus_{x \in X} \oplus_{j=1}^{m(x)} (\mathbb{Z}/d)^2.$$

Example 2.31. If we take $[X_{x,j}]$ and look at its class in the quotieint, it becomes $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ in the (x, j)-th summand.

Definition 2.32. Observe that $(\mathbb{Z}/d)^2$ has a natural symplectic form given by $J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, we put that on each block in

$$\mathrm{P}(X, m) \coloneqq \mathcal{P}(X, m) / \operatorname{Center}(\mathcal{P}(X, m)) \cong \bigoplus_{x \in X} \oplus_{j=1}^{m(x)} (\mathbb{Z}/d)^2.$$

Now we define a symplectic form

$$\Omega : \mathrm{P}(X, m) \times \mathrm{P}(X, m) \to \mathbb{Z}/d$$

given as the map

$$\sum_{x,j} \begin{pmatrix} a & b \end{pmatrix} J \begin{pmatrix} c \\ d \end{pmatrix}$$

Observation: Let $\alpha : \mathcal{P}(X, m) \to \mathcal{P}(X, m)$ be a Clifford QCA with constant ℓ, then we also have a map over the quotients

$$\bar{\alpha} : \mathrm{P}(X, m) \to \mathrm{P}(X, m).$$

that is a symplectictomorphism. Furthermore, there is a locality preserving condition that

$$(\overline{\alpha})^{y,k}_{x,j} : \mathrm{P}_{x,j} \to \mathrm{P}_{y,k} = 0 \text{ if } d(x,y) > \ell.$$

The upshot is that we have turned Clifford QCA to a question about abelian group maps preserving the symplectic structure.

Definition 2.33. The collection of locality-preserving bijective symplectomorphisms over $\mathrm{P}(X,m)$ is denoted as $\mathrm{Sp}_{loc}(X,m)$.

Theorem 2.34. The group of Clifford QCAs over (X,m), denoted $\mathcal{Q}^{\mathrm{Cliff}}(X,m)$, admits a surjection

$$\mathcal{Q}^{\mathrm{Cliff}}(X,m) \to \mathrm{Sp}_{loc}(X,m), \quad \alpha \mapsto \bar{\alpha}.$$

There is also a surjection

$$\mathcal{Q}^{\mathrm{Cliff}}(X,m) \to \mathrm{CQ}(X,m),$$

where $\mathrm{CQ}(X,m)$ is the Clifford QCA group modulo the Clifford circuits (the subgroup generated by single-layer circuits that are Clifford). Together, there is a lift, with all arrows being surjection, as:

Theorem 2.35.

$$\begin{array}{ccc} \mathcal{Q}^{\mathrm{Cliff}}(X,m) & \twoheadrightarrow & \mathrm{CQ}(X,m) \\ \downarrow & \nearrow & \\ \mathrm{Sp}_{loc}(X,m) & & \end{array}$$

Taking colimit over m, we get a very similar classification group $\mathrm{CQ}(X)$ and also $\mathrm{Sp}_{loc}(X)$. This gives a surjection

$$\mathrm{Sp}_{loc}(X) \to \mathrm{CQ}(X).$$

Remark 2.36. These all depend on the choice of d, but we suppress them in notation.

Yesterday we mentioned that there is a surjection

$$\mathrm{Sp}_{loc}(X) \to L_3(X, \mathbb{Z}/d),$$

where $L_3(X,\mathbb{Z}/d)$ is Pedersen-Weibel's bounded even symmetric L-theory. It turns out that this completes to a factorization of surjections:

Theorem 2.37.

$$\begin{array}{ccc} & \mathrm{CQ}(X,m) & \\ \nearrow & & \overset{\pi}{\searrow} \\ \mathrm{Sp}_{loc}(X,m) & \twoheadrightarrow & L_3(X,\mathbb{Z}/d) \end{array}$$

Theorem 2.38 (Y. in [Yan26a])**.** When $X = \mathbb{Z}^n$ and $n \geq 2$, the kernel of π is 0.

Remark 2.39. We only really care about Clifford QCA for $n \geq 2$, since for $n = 1$ the usual QCAs are completely classified.

We can check that, from L-theory literature, that

Theorem 2.40. $L_3(\mathbb{Z}^n; \mathbb{Z}/d) \simeq L^?_{3-n}(\mathbb{Z}/d)$.
Caveat: There are multiple L-theories when you go away from the bounded even symmetric L-theory, but this problem only happens when d is a power of 2.

When $d = p^a$ and p is odd, we have that:

1. If $n \equiv 3 \pmod 4$, then
$$\mathrm{CQ}(\mathbb{Z}^n) = \begin{cases} \mathbb{Z}/2 \times \mathbb{Z}/2, p \equiv 1 \pmod 4 \\ \mathbb{Z}/4, p \equiv 3 \pmod 4 \end{cases}$$
2. If $n \not\equiv 1 \pmod 4$, then
$$\mathrm{CQ}(\mathbb{Z}^n) = 0.$$

When $d = 2^a$ and $n \geq 6$, we have that

$$\mathrm{CQ}(\mathbb{Z}^n) = \begin{cases} \mathbb{Z}/2, n \text{ is even} \\ 0, n \text{ is odd} \end{cases}$$

The unstable part of Grothendieck-Witt groups ($n < 6$) was computed by M. Nicolas [Nic25]:

	$d = 2$	$d = 4$	$d = 2^a, a \geq 3$
$n = 3$	$\mathbb{Z}/2$	$\mathbb{Z}/8$	$\mathbb{Z}/8 \oplus \mathbb{Z}/2$
$n = 4$	0	0	0
$n = 5$	$\mathbb{Z}$	?	?

where ? marks were either $\mathbb{Z}/2$ or 0.

3 Lecture 3 (August 26th, 2026)

3.1 The QCA Spectrum and the Delooping Theorem

Today we will first talk about the actual QCA theory (as in, the full Ω-spectrum), following [JY26b]. In the second half, we will talk about some applications of theory to global symmetries, following [JY26a].

Again, we recall the following notations:

1. We let X be a (discrete or nice) metric space (e.g., $\mathbb{Z}^n$).
2. R be a commutative ring with identity.
3. A **quantum spin system** $q : X \to \mathbb{Z}_{>0}$ specifies the dimension on each point.
4. The **algebra of observables** $\mathcal{A}(X, q)$ is, roughly speaking, the tensor product $\bigotimes_{x\in X} M_{q(x)}(R)$.

3.1.1 The QCA Category

Today we will assemble that $\mathcal{A}(X, q)$'s into a category.

Definition 3.1. The **category of quantum spin systems** (or the **QCA category**), denoted $\mathbf{C}(X; R)$, has

1. **Objects** are $\mathcal{A}(X, q)$ for all quantum spin systems q.
2. **Morphisms** are locality-preserving R-algebra isomorphisms (this can go between two different objects). This is an R-algebra isomorphism
$$\alpha : \mathcal{A}(X, q) \to \mathcal{A}(X, r)$$
such that there exists some $\ell \geq 0$, such that for any $A \in \mathcal{A}(X, q)$, we have
$$\operatorname{supp} \alpha(A) \subseteq \ell\text{-neighborhood of } \operatorname{supp}(A).$$
3. Furthermore, we have a **symmetric monoidal operation** as $\mathcal{A}(X, q) \otimes \mathcal{A}(X, r) = \mathcal{A}(X, q \cdot r)$.

Remark 3.2. In the unitary case over $\mathbb{C}$, we denote the unitary QCA category as $\mathbf{C}^*(X)$.

Example 3.3. In the definition above, the morphisms need not be automorphisms. For example, over $\mathbb{Z}^1$, we can have the following two quantum spin systems:
$$q(n) = \begin{cases} 2, n \text{ is odd} \\ 3, n \text{ is even} \end{cases} \quad \text{and } r(n) = \begin{cases} 3, n \text{ is odd} \\ 2, n \text{ is even} \end{cases}.$$
There is clearly a locality preserving isomorphism between them that swaps the tensor factors.

Let us look at some examples of the QCA category for concreteness.

Example 3.4.
1. When $X = *$, $\mathbf{C}(*; R) = \mathrm{Mat}_R$, the symmetric monoidal category of matrix rings over R, with only R-algebra automorphisms and the symmetric monoidal operation of tensor product.
2. When $X = \{p, q\}$ is a discrete metric space with two points. $\mathbf{C}(X; R)$ is equivalent to $\mathbf{C}(*; R)$ induced by folding the two points together.

More generally, the category $\mathbf{C}(X;R)$ is invariant under **coarse equivalences** of the input space X. This reflects the principle that it only depends only on the large-scale (coarse) geometry of X (i.e., robustness against dirt).

Our next step was to consider the algebraic K-theory of the QCA category.

Definition 3.5. Consider the K-theory of the category $\mathbf{C}(X;R)$.

$$K(\mathbf{C}(X;R)) = (B\mathcal{C}(X;R))^{gp}.$$

Note: We sometimes call the space $K(\mathbf{C}(X;R))$ as the **space of quantum spin systems**.

We show that $K(\mathbf{C}(X;R))$ recovers the QCA classification group as follows.

Proposition 3.6 (Ji-Y. [JY26b]). There is an isomorphism

$$\pi_1(K(\mathcal{C}(X;R)) \cong \mathcal{Q}(X;R)^{ab}.$$

Remark 3.7. Recall from lecture 2, the statement that "$\mathcal{Q}(X;R)^{ab}$ is equal to QCA over circuits" is true in almost all cases of interest. This is true when (1) R is an algebraically closed field or (2) $X = Y \times \mathbb{Z}$. We elaborate on this more in Appendix A.

We also investigated K_0 of this category.

Theorem 3.8 (Ji-Y. [JY26b]). For $n \geq 1$ we have that $\pi_0 K\mathcal{C}(\mathbb{Z}^n;R) = 0$.

3.1.2 The QCA Space and Delooping

Since we would like π_0 of our space to classify QCA, we can take an additional loop on the K-theory space of the QCA category to line up the π_0. This motivates the following definition of **the space of QCA**.

Definition 3.9. We define the **space of QCA** over $(X;R)$ to be

$$\mathbf{Q}(X;R) \coloneqq \Omega K(\mathbf{C}(X;R)).$$

Since we have taken a based loop space, we now have that

$$\pi_0\mathbf{Q}(X;R) = \mathcal{Q}(X;R)^{ab}.$$

Remark 3.10. Typically taking a (based) loop space discards the π_0-information, but Theorem 3.8 tells us that we do not lose any information when we take a loop.

Now we discuss the main theorem of [JY26b].

Theorem 3.11 ([JY26b]). There is an equivalence

$$\Omega\mathbf{Q}(\mathbb{Z}^{m+1};R) \simeq \mathbf{Q}(\mathbb{Z}^m;R).$$

In other words, the following collection of spaces form an Ω-spectrum $\{\mathbf{Q}(\mathbb{Z}^m;R)\}_{m\geq 0}$. Furthermore, there

is a stronger statement that for nice Y,

$$\Omega\mathbf{Q}(Y \times \mathbb{Z}; R) \simeq \mathbf{Q}(Y; R).$$

Remark 3.12. Analogous results hold for the unitary case over $\mathbb{C}$.

This shows us that QCA spaces form a sequence of delooping indexed by the lattice dimension. This gives us the data of an Ω-spectrum. To quickly recall what that means:

Definition 3.13. An Ω-spectrum E is a sequence of spaces $E_0, E_1, E_2, \ldots$ with the data of weak equivalences $\Omega E_{n+1} \simeq E_n$ for $n \geq 0$.

As a consequence of Theorem 3.11, we can define the K-theoretic QCA spectrum as follows.

Definition 3.14. For any ring R, the **QCA spectrum** $\mathbb{QCA}(R)$ over R is the sequence of spaces

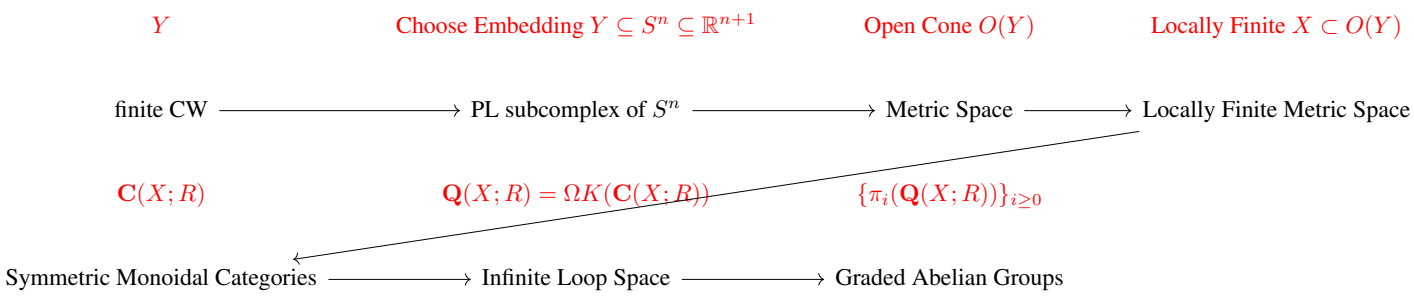
$$\{\mathbf{Q}(\mathbb{Z}^n; R)\}_{n\geq 0}$$

with the data of the canonical deloopings $\Omega\mathbf{Q}(\mathbb{Z}^{n+1}; R) \simeq \mathbf{Q}(\mathbb{Z}^n; R)$ from Theorem 3.11.

More generally, for any nice metric space X, we can define an Ω-spectrum $\mathbb{QCA}(X; R)$ as $\{\mathbf{Q}(X \times \mathbb{Z}^n; R)\}_{n\geq 0}$ with the data of canonical deloopings. This recovers $\mathbb{QCA}(R)$ when X is a single point.

3.1.3 Proof of the Delooping Theorem

Before we start giving a proof sketch for the theorem, let us first discuss how this gets back to the open cone discussion in Lecture 1. This is because we now have the following flowchart:



Remark 3.15. If there is an inclusion $X \to X'$, this induces a lax symmetric monoidal functor

$$\mathcal{C}(X; R) \to \mathcal{C}(X'; R)$$

by sending $\mathcal{A}(X, q) \to \mathcal{A}(X', q')$ where

$$q'(x) = \begin{cases} q(x), x \in X \\ 1, x \notin X \end{cases}.$$

More generally, the QCA spaces here are functorial with respect the **proper and eventually Lipschitz** maps of metric spaces.

Let us now explain the proof ideas behind the delooping theorem.

Proof Idea. It would be great to show that

$$\Omega K(\mathbf{C}(\mathbb{Z}^{m+1}; R)) \simeq K(\mathbf{C}(\mathbb{Z}^m; R)).$$

But this is very unlikely to be true: the RHS has trivial π_0 for $m \geq 1$ and the LHS has π_0 being the abelianization of the total QCA group. This would imply all QCAs are circuits, but this is clearly defeating the purpose of the field.

Recall: Last time we noted that

$$\pi_i K(\mathrm{Free}(R)^{\cong}) = \pi_i K(\mathrm{Proj}(R))^{\cong}, i \geq 1$$

but the isomorphism is not true at $i = 0$.

Observation: What went wrong was on the level of π_0.

In [PW85], they were able to resolve their π_0 issue using an operation called **idempotent completion** (which informally means we can get projection maps). However, this is very not applicable to our scenario, because in general there are no canonical projections from $A \otimes_R B$ back to A.

Instead, we resorted to **enlarging the category by hand**, using an approach inspired by Haah in [Haa23] adapted to our setting.

Definition 3.16 (Haah [Haa23], adapted to our context)**.** A unital R-subalgebra $B \subseteq \mathcal{A}(X, q)$ is **invertible** if there exists $\ell > 0$ such that for any $A \in \mathcal{A}(X, q)$, we can write A as a finite sum

$$A = \sum_i B_i C_i$$

such that $\operatorname{supp} B_i, \operatorname{supp} C_i \subseteq$ the ℓ-neighborhood of $\operatorname{supp}(A)$, the $B_i \in B$, $C_i \in B' \coloneqq \{c \in \mathcal{A}(X, q) \mid 0 = [c, b], \forall b \in B\}$.

Remark 3.17. This is mimicking Azumaya algebras / projective modules.

We write $\mathbf{C}'(X; R)$ to denote the symmetirc monoidal category of all invertible subalgebras, where $\mathbf{C}(X; R)$ is cofinal within $\mathbf{C}'(X; R)$. Since the based loop is taking tat the identity, you can alternatively say

$$\mathbf{Q}(X; R) \coloneqq \Omega K(\mathbf{C}'(X; R))$$

The corrected theorem that we did prove is:

Theorem 3.18. $\Omega K(\mathbf{C}'(\mathbb{Z}^{n+1}; R)) \simeq K(\mathbf{C}'(\mathbb{Z}^n; R))$.

The core idea of the proof used Thomason's **double mapping cylinder construction** in [Tho82]. For symmetric monoidal categories A, B, C and lax symmetric monoidal functors $u : A \to B, v : A \to C$, it is possible to get a homotopy pullback:

$$\begin{array}{ccc} K(A) & \xrightarrow{u_*} & K(B) \\ {\scriptstyle v_*}\downarrow & & \downarrow \\ K(C) & \longrightarrow & K(P) \end{array}$$

where P is a symmetric monoidal category depending on A, B, C.

Now we apply this to $A = \mathbf{C}'(\mathbb{Z}^m; R)$, $B = \mathbf{C}(\mathbb{Z}^m \times \mathbb{Z}_{\geq 0}; R)$, and $C = \mathbf{C}(\mathbb{Z}^m \times \mathbb{Z}_{\leq 0}; R)$. This gives us the diagram

$$\begin{array}{ccc} K(\mathbf{C}'(\mathbb{Z}^m; R)) & \xrightarrow{u_*} & K(\mathbf{C}'(\mathbb{Z}^m \times \mathbb{Z}_{\geq 0}; R)) \\ \downarrow{\scriptstyle v_*} & & \downarrow \\ K(\mathbf{C}'(\mathbb{Z}^m \times \mathbb{Z}_{\leq 0}; R)) & \longrightarrow & K(P) \end{array}$$

where the initial maps are induced by the natural inclusion of subspaces.

Lemma 3.19. $K(\mathbf{C}'(\mathbb{Z}^m \times \mathbb{Z}_{\geq 0}; R))$ and $K(\mathbf{C}'(\mathbb{Z}^m \times \mathbb{Z}_{\leq 0}; R))$ are both contractible.

Proof Idea of Lemma. Eilenberg Swindle. ■

Using this lemma, we have now:

$$\Omega K(P) \simeq K(\mathbf{C}'(\mathbb{Z}^m; R)).$$

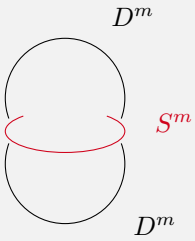
It remains to show that $K(P) \simeq K(\mathbf{C}'(\mathbb{Z}^{m+1}; R))$, which follows from Quillen's Theorem A (see [Qui73] and Theorem IV.3.7 of [Wei13]) and a lot of non-commutative algebra. Note that the category P is in general not equivalent to $\mathbf{C}'(\mathbb{Z}^{m+1}; R)$. ■

Remark 3.20. The proof can be adapted similarly to show that in nice cases for U, V (satisfying our standing assumptions), we have a homotopy pullback

$$\begin{array}{ccc} \mathbf{Q}(U \cap V; R) & \longrightarrow & \mathbf{Q}(V; R) \\ \downarrow & & \downarrow \\ \mathbf{Q}(U; R) & \longrightarrow & \mathbf{Q}(U \cup V; R) \end{array}$$

as taking a based loop is right adjoint. Certainly we can use Thomason's theorem to get a square to match three corners, and a similar argument with Quillen's Theorem A will get the remaining corner.

Remark 3.21. Recall when studying vector bundles over S^m, we can look at its restriction to the upper and lower hemisphere:



The vector bundle is trivial over either hemisphere, and it is determined by a clutching function on S^m.

Similarly, for QCA's we see somehow the spaces are trivial on $\mathbb{Z}^m \times \mathbb{Z}_{\geq 0}$ and $\mathbb{Z}^m \times \mathbb{Z}_{\leq 0}$, and we see that invertible subalgebras are really the clutching functions on $\mathbb{Z}^n$.

3.2 Application To Symmetries and Anomalies

We conclude this lecture by discussing some applications of the QCA theory described in [JY26b]. This is the content of [JY26a].

Assumption: For simplicity, we work in the unitary case over $\mathbb{C}$, but the results are generalized over any field in the actual paper.

3.2.1 K-Theoretic Obstruction Theories for Onsiteability

It is natural to study symmetries of quantum many-body systems (i.e., $\mathcal{A}(X,q)$) with respect to a group G. As QCAs are models for the dynamics of quantum many-body systems, one way to model these symmetries is to consider a **group homomorphism**

$$\varphi : G \to \mathrm{Aut}_{\text{locality-preserving}}(\mathcal{A}(X,q)) = \mathcal{Q}(X,q).$$

Let us give this a name.

Definition 3.22. A **QCA representation** of a group G is a group homomorphism

$$\varphi : G \to \mathcal{Q}(X,q).$$

In the study of QCAs and anomalies, one common question is whether or not a given QCA representation would restrict to a somewhat trivialized form of representation. More precisely:

Question 3.23. Does there exist $\beta \in \mathcal{Q}(X,q)$, such that the following diagram lifts

$$\begin{array}{ccc} & & \prod_{x\in X} U_{q(x)} \\ & \nearrow & \downarrow \\ & & \prod_{x\in X} PU_{q(x)} \\ & \nearrow & \downarrow \\ G & \xrightarrow{\beta\varphi\beta^{-1}} & \mathcal{Q}(X,q) \end{array} \tag{1}$$

Remark 3.24. In the paper [JY26a], we referred to "onsiteable" as **linearizable** and "stably onsiteable" as **stably linearizable** to emphasize that the lift in Question 3.23 lifts past the projective part, and hence is in the linear part.

Definition 3.25. We say a QCA representation $\varphi : G \to \mathcal{Q}(X,q)$ is

1. **onsiteable** (e.g., anomaly free, or gaugeable) if φ fits into (1) for some β.
2. **stably onsiteable** if $\varphi \otimes 1_r : G \to \mathcal{Q}(X, q\cdot r)$ for some r. Here $1_r : G \to \mathcal{Q}(X,r)$ is the constant map to identity.

In [JY26a], we used the algebraic K-theory spectrum/space constructed prior to give homotopical and cohomological obstructions for being (stably) onsiteable.

Theorem 3.26 (Ji-Y.)**.** For G finite, if φ is stably onsiteable, then there is an induced map Φ, depending on φ,

$$\Phi : BG \to K\mathbf{C}^*(X) \text{ that is null-homotopic.}$$

Here, $\mathbf{C}^*(X)$ is our notation for the unitary QCA category.

As a corollary, we can sequentially extract these cohomology classes using a universal construction called Dror's tower (originally developed in Dror's thesis [Dro72]). This allows us to develop a series of universal cohomological obstruction classes as follows.

Corollary 3.27. When $X = \mathbb{Z}^n$ and G is finite, there is a complete list of cohomological obstruction classes that obstruct the null-homotopy of Φ, living in

1. $u_1(\varphi) \in H^1(BG; \mathcal{Q}^*(\mathbb{Z}^n)^{ab})$.
2. $u_2(\varphi) \in H^2(BG; \mathcal{Q}^*(\mathbb{Z}^{n-1})^{ab})$.
3. $u_3(\varphi) \in H^3(BG; \mathcal{Q}^*(\mathbb{Z}^{n-2})^{ab})$.
4. ...
5. $u_n(\varphi) \in H^n(G; \mathcal{Q}^*(\mathbb{Z}^0))^{ab})$.
6. $u_{n+1}(\varphi) \in H^{n+1}(BG; U(1))$

Here we recall that $\mathcal{Q}^*(\mathbb{Z}^k)$ refers to the unitary QCA group, and its abelianization is the unitary QCA classification group.

3.2.2 Low Degree Obstruction Classes

Here, we explain briefly what some **low degree obstruction classes** are concretely.

Degree-1: Observation that any **onsiteable** QCA representation is a **circuit**. Thus, a **stably onsiteable** has to be **stably a circuit** (it is tensoring with a circuit to begin with). In other words, the map

$$\varphi : G \to \mathcal{Q}^*(X, q) \to \mathcal{Q}^*(X)$$

must land in the circuit subgroup $\mathcal{C}^*(X) \subseteq \mathcal{Q}^*(X)$.

Definition 3.28. We can view the detection for whether or not φ lands in the circuit subgroup as an element in

$$H^1(BG, \mathcal{Q}^*(X)/\mathcal{C}^*(X)) = H^1(BG; \mathcal{Q}^*(X)^{ab}).$$

Proposition 3.29 (See [JY26a]). The universal obstruction $u_1(\varphi)$ in Corollary 3.27 agrees with Definition 3.28.

The next construction will only be discussed over the integer lattices.

Degree-2: Let us suppose we have a QCA representation φ that **passed the degree-1 test**, that is, it is of the form

$$\varphi : G \to \mathcal{C}^*(\mathbb{Z}^n)$$

and lands in the circuit subgroup already. We need a test at one degree higher now to test this. The following construction class is inspired by [KS25].

Definition 3.30 (Adapted from [KS25]). Let $H = \mathbb{Z}^{n-1} \times \{0\}$ be an oriented hyperplane in $\mathbb{Z}^n$ (with identified positive and negative side, wrriten as H^+ and H^-). We can consider a truncation operation

$$t_+ : \mathcal{C}^*(\mathbb{Z}^n) \longrightarrow \mathcal{C}^*(H^+) \tag{2}$$

by choosing a circuit representative as finite depth-circuits, and then retain only the blocks in H^+. For $\alpha, \beta \in \mathcal{C}^*(\mathbb{Z}^n)$, we can consider the term

$$t_+(\alpha)t_+(\beta)t_+(\alpha\beta)^{-1}. \tag{3}$$

This acts as the identity sufficiently far away from $\mathbb{Z}^{n-1} \times \{0\}$, thus it collapses to a QCA 1-dimension lower.

We have now specified a cocycle

$$\mathcal{C}^*(\mathbb{Z}^n) \times \mathcal{C}^*(\mathbb{Z}^n) \to \mathcal{Q}^*(\mathbb{Z}^{n-1})/\mathcal{C}^*(\mathbb{Z}^{n-1}),$$

which we can compose with φ to get a 2-cocycle

$$\tau : G \times G \to \mathcal{Q}^*(\mathbb{Z}^{n-1})/\mathcal{C}^*(\mathbb{Z}^{n-1})$$

This gives a **cohomology class**

$$[\tau] \in H^2(BG; \mathcal{Q}^*(\mathbb{Z}^{n-1})/\mathcal{C}^*(\mathbb{Z}^{n-1})) = H^2(BG; \mathcal{Q}^*(\mathbb{Z}^{n-1})^{ab}).$$

It turns out this cohomology class is independent of prior choices made.

Theorem 3.31 (See Appendix A of [JY26a])**.** The class τ defined above is a model for the second universal obstruction class $u_2(\varphi)$ in Corollary 3.27.

Remark 3.32. More generally, this construction can be adapted with respect to any oriented hyperplane H in $\mathbb{R}^n$, not necessarily the present one described here.

3.3 The Analogies between Algebraic K-theory and QCA

We conclude the main notes with a table of analogies.

Algebraic K-theory	Quantum Cellular Automata
Free R-Modules R^n	Matrix R-Algebras $M_n(R)$
Projective R-Modules	Azumaya R-Algebras
Direct Sum $\oplus$	Tensor Product $\otimes$
Reduced $\widetilde{K}_0(R)$	Brauer Group $\mathrm{Br}(R)$
Negative K-theory	QCA Groups
The spectrum $\mathbb{K}(R)$	The spectrum $\mathbb{QCA}(*; R)$
Bounded K-theory	$\mathbb{QCA}(X; R)$

Open Question 3.33. What can we say about **QCA groups** using **techniques from K-theorists**? What can we say about **multiplicative K-theory** using **techniques from the quantum physicists**?

A Models of Quantum Circuits

It turns out that there are several reasonable choices to generalize that all become equivalent over $\mathbb{C}$ (or in reasonable circumstances), which we will elaborate.

Definition A.1. Let (X, R, q) be given as before. A **single-layer circuit** is:

1. A choice of partition of $X = \bigsqcup_j X_j$ such that $\mathrm{diam}(X_j)$ is uniformly bounded with respect to j.
2. A choice of $\alpha_j \in \mathrm{Aut}_R \mathcal{A}(X_j, q)$ for each j.
3. This gives a QCA on X of the form
$$\alpha := \bigotimes_j \alpha_j.$$

Write $q(X_j) := \prod_{x \in X_j} q_x$. We say a single-layer circuit α is

1. **Inner** if $\alpha_j \in \mathrm{PGL}_{q(X_j)}(R)$ for each j.
2. **Special Linear** if $\alpha_j \in \mathrm{PSL}_{q(X_j)}(R)$ for each j.
3. **Elementary** if α_j is conjugation of an elementary matrix E_j, for each j.

The subgroup of $\mathcal{Q}(X; R)$ generated by all single-layer circuits, all single-layer inner circuits, all single-layer special linear circuits, and all single-layer elementary circuits are denoted

$$\mathcal{C}(X; R) \supseteq \mathcal{C}^{\mathrm{inn}}(X; R) \supseteq \mathcal{C}^{\mathrm{sp}}(X; R) \supseteq \mathcal{C}^{\mathrm{el}}(X; R).$$

It is an algebraic exercise to check that:

Proposition A.2 (See Section 2 of [JY26b]). The groups $\mathcal{C}^{\mathrm{el}}(X; R), \mathcal{C}^{\mathrm{sp}}(X; R), \mathcal{C}^{\mathrm{inn}}(X; R), \mathcal{C}(X; R)$ are all **normal subgroups** of $\mathcal{Q}(X; R)$. Furthermore, the quotient of $\mathcal{Q}(X; R)$ by each subgroup are all abelian.

Proof Sketch. Let $\beta \in \mathcal{Q}(X, q)$ be a QCA of spread ℓ, and α be a circuit of any type with

$$\alpha = \bigotimes_j \alpha_j \text{ for } X = \bigsqcup_{j \in J} X_j.$$

We can view α_j as an automorphism of X that acts as identity outside of X_j, and hence we have that $\beta \circ \alpha_j \circ \beta^{-1}$ is a QCA over X_j^ℓ (the ℓ-neighborhood of X_j, outside of this the β and its inverse cancel). The term $\beta \circ \alpha_j \circ \beta^{-1}$ would remain to be a circuit of the same type over X_j^ℓ (as each respective group is normal in $\mathrm{Aut}_R(M_n(R))$).

Thus, we can write

$$\beta \circ \alpha \circ \beta^{-1} = \bigotimes_j \beta \circ \alpha_j \circ \beta^{-1}.$$

The caveat is that each $\beta \circ \alpha_j \circ \beta^{-1}$ is no longer an automorphism of a disjoint partition of X, and their supports can overlap. This is not an issue, however, if each point only has only finitely many $\beta \circ \alpha_j \circ \beta^{-1}$ overlapping over. In this case, we can rearrange so that operators with disjoint support are assigned to the same layer. This geometric condition is guaranteed because **we assumed X has bounded geometry**.

This shows that each subgroup is normal. To show that their quotients are abelian, it suffices to show that the quotient $\mathcal{Q}(X; R)/\mathcal{C}^{\mathrm{el}}(X; R)$ is abelian. This follows from an algebraic calculation using a QCA called $\mathrm{SWAP}_{q,r}$ that interchanges $\mathcal{A}(X, q) \otimes \mathcal{A}(X, r)$ with $\mathcal{A}(X, r) \otimes \mathcal{A}(q)$. It is a consequence of linear algebra manipulation (see for example [LTK+24]) that these swap QCA are actually elementary quantum circuits . ■

As a consequence, we now have an inclusion of normal subgroups of $\mathcal{Q}(X;R)$ as:

$$[\mathcal{Q}(X;R),\mathcal{Q}(X;R)] \subseteq \mathcal{C}^{\mathrm{el}}(X;R) \subseteq \mathcal{C}^{\mathrm{sp}}(X;R) \subseteq \mathcal{C}^{\mathrm{inn}}(X;R) \subseteq \mathcal{C}(X;R).$$

What is quite surprising is that all groups in the sequence actually agree in many reasonable cases.

Theorem A.3 (See Corollary C of [JY26b]). Suppose we have either one of the following cases:

(a) $R = k$ be a field saistifying $(k^\times)^n = k^\times$ (e.g., algebraically closed field)

(b) $X = Y \times \mathbb{Z}$

It follows that every circuit group is equal to the commutator $[\mathcal{Q}(X;R),\mathcal{Q}(X;R)]$.

Proof Sketch. For (a), it can be shown, using the generalized Whitehead lemma (Lemma 1.8 on page 351 of [Bas68]), that $[\mathcal{Q}(X;R),\mathcal{Q}(X;R)]$ is the maximal perfect normal subgroup of $\mathcal{Q}(X;R)$. By Proposition A.2, it suffices to check all subgroups are perfect. It suffices to check that all singe-layer circuits α are products of commutators.

Since $(k^\times)^n = k^\times$, we know that the automorphism group is isomorphic to the projective general linear group, which is also isomorphic to the projective special linear group, and all elementary matrices here are special linear. It is a general fact that $\mathrm{PGL}_n(\mathbb{F})$ is perfect for a field $\mathbb{F}$ and n sufficiently large. Thus, we know each α_j is a product of commutators. It remains to show that each element can be written as a product of commutators with uniformly bounded length. Theorem 1 and 2 of [Tho61] shows that this is true, and the length can be in fact taken to be 2.

Remark A.4. The same proof also shows that $\mathcal{C}^{sp}(X;\mathbb{F})$ is the commutator subgroup of $\mathcal{Q}(X;\mathbb{F})$ for any field $\mathbb{F}$.

For (b), suppose we have that $X = Y \times \mathbb{Z}$. It suffices to show that a single-layer circuit of any type is trivial in the abelianization. We can clearly break a single-layer circuit α into two parts $\alpha^+ \otimes \alpha^-$ where α^+ restricts to identity on the $X \times \mathbb{Z}_{\leq \ell}$ for sufficiently low ℓ, and α^- restricts to identity on $X \times \mathbb{Z}_{\geq \ell}$ for sufficiently high ℓ.

The proof now follows from a lemma that the abelianization of $\mathcal{Q}(X \times \mathbb{N})$ is trivial, which follows from an argument similar to **Eilenberg swindle**. ■

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